D The shadow price of capital and the weighted average discount rate

The shadow price of an output or input measures all its social costs and benefits. Just as a businessman evaluates a project by its prospective profit, the government can evaluate a project by its increase in efficiency. Profit measures the extra consumption the project generates for its owner. The net present value of a government project measures the extra consumption it generates for society. Profit is calculated by assessing all the project inputs and outputs relevant to the project owner and converting these into costs and revenues using market prices. These are the prices paid by the business and determine profits. The 'social profit' (or efficiency effects) of a government project are calculated by assessing all of the project inputs and outputs for society and converting these into costs and benefits using shadow prices, which reflect the social value of project inputs and outputs.

Shadow prices differ from market prices when taxes, subsidies, externalities, monopoly, and price and quantity controls distort markets.[1] For example, when a government project uses a taxed input, there is a standard technique in cost benefit analysis for deriving its shadow price (that is, determining its social value).[2] When an input is taxed, the tax drives a wedge between its after-tax supply price and its before-tax demand price. When a government project demands extra units of the input, its price increases to decrease private demand and increase supply until the market satisfies the extra demand. The social cost of the extra input depends on whether it mainly decreases private consumption of the good or increases private supply. That part of government input use that comes from increased private supply is valued at the supply price, that which decreases consumption at the demand price. The shadow price, or social value, of the output is a weighted average of the demand and supply price — with the displacement of private demand and supply determining the weights.

Another way to look at it is that the supply price is the social cost of providing extra units of input the project uses. But when demand is reduced, that decreases tax

1 See Kanbur (2008).

2 See, for example, Department of Finance and Administration (2006, pp. 35-36). For more detail, see Harberger (1971).

revenue — a social cost, increasing the social cost of the extra input above the (after-tax) supply price.

Capital taxes mean that forgone private investment has a cost that is not reflected in project cost and benefit flows. \$1 of private investment has a social value greater than \$1. When determining the social value of a government project which gives costs and benefits over time, we need to shadow price the capital it uses. That is, we account for the consumption that forgone private investment would have produced, adding it to project costs to reflect the full impact of the project on consumption and discount with the consumption rate.

It is more complicated than the standard input case outlined above, as capital lasts for multiple periods and depreciates. The case considered here is a closed economy where all capital returns are taxed at a single rate.

In a closed economy, government investment must come at the expense of consumption or private investment. The shadow price approach calculates the social value of all consumption and investment impacts, converting them all to consumption equivalents. The first step is to determine whether private investment flows will be altered by a policy. Changes in investments are converted into equivalent units of consumption. All flows of consumption and consumption–equivalents are then discounted using the (after-tax) consumption rate of interest, the rate consumers would use in discounting future consumption benefits.

It turns out that discounting ordinary cost and benefits flows using a weighted average discount rate (usually) gives the same answer.

D.1 A perpetuity

Consider a one unit public investment that generates a perpetuity with a real rate of return δ. As the investment is a current cost borne for future benefits, it uses a unit of capital. The capital gives a flow of consumption benefits of δ per year forever. A fraction a of the investment takes place at the expense of private investment and a fraction $(1-a)$ at the expense of consumption (see appendix B for how these fractions are determined).

If private investment produced a perpetual real return i per year and the consumption is discounted at rate r, with a tax rate t on the real return so that $r = (1-t)i$. Then the present value of the consumption produced by \$1 of private capital is:

$$i/(1+r) + i/(1+r)^2 + i/(1+r)^3 + \ldots = i/r = 1/(1-t) .$$

which is greater than \$1. One dollar of the capital stock produces a stream of annual consumption benefits worth i cents per year, with $r = (1 - t)i$ cents going to the owner and $(i - r) = ti$ cents in tax payments to the government. The present value to the owner is the value of r cents forever discounted at r per cent, which is $\$r/r = \$1$. The present value of the tax revenues is: $\$(i - r)/r = \$(i/r - 1)$. The present value of the benefits to society from an extra unit of private investment is the sum of these two: $\$(i/r - 1) + \$1 = \$\, i/r$.

The project displaces private investment that would have produced ai dollars per year. The social cost of the project is the present value of forgone consumption (discounting the consumption flows with the consumption interest rate):

$$(1-a)+ai/(1+r)+ai/(1+r)^2+ ai/(1+r)^3+\ldots = (1-a) +ai/r.$$

This is the shadow price of capital used in the project, its opportunity cost. It is the value of the consumption that the capital used in the project would have produced. That is, the project produces output (a benefit), but uses resources, reducing private sector output (a cost).

The project improves efficiency if its benefits exceeds its costs. That is, if the present value of consumption the project produces is greater than its cost

$$\delta/(1+r)+\delta/(1+r)^2+ \delta/(1+r)^3+\ldots.= \delta/r \geq (1-a) +ai/r. \text{ Or}$$

$$\delta \geq (1-a)r +ai = w$$

The project is worthwhile if the return on investment is greater than the weighted average discount rate, where the weights depend on the amount of investment and consumption the project displaces. In other words, if the project has a positive net present value using a discount rate w:

$$\delta/(1+w)+\delta/(1+w)^2+ \delta/(1+w)^3+\ldots.= \delta/w \geq 1$$

It generates enough benefits to more than compensate private investors and consumers for their forgone consumption. With perpetuities, the shadow pricing approach and weighted average discount rate are two equivalent ways to work out the opportunity cost of capital used in a project.

D.2 In a two period world

The same is true in a two period world. Take a government project that costs 1 unit in period 1 and produces $1+\delta$ of benefits in period 2 and displaces $(1-a)$ units of consumption and a units of investment. The displaced private investment would have produced $a(1+i)$ of benefits in period 2.

The project is worthwhile if:

$$(1+\delta)/(1+r) \geq (1-a) + a(1+i)/(1+r)$$

The left hand side is the present value of the period 2 consumption benefits the project produces. The right hand side is the shadow price of the capital used in the project, its opportunity cost – the forgone consumption in period 1 and the present value of the forgone consumption in period 2 from displaced private investment. The above project acceptance criterion reduces to:

$$\delta \geq (1-a)r + ai = w$$

The project is efficient if its rate of return exceeds the weighted average discount rate, w. Again the shadow pricing approach and weighted average approach are equivalent.

D.3 In a multi-period world

In a multi-period world, we need to worry about re-investment of project output. As the present value of consumption from $1 of investment is greater than $1, the amount of re-investment will affect the value of the project.

For example, consider the appropriate discount rate for a two period project in multi-period world. This is quite general as multi-period projects can be expressed as a sequence of two period projects. For example, an investment that pays off in two periods time is equivalent to an investment that pays off next period combined with a project which reinvests that output and pays off the following period.

One unit is invested in the current period, which produces output $(1+\delta)$ in the following period 2. The project is sourced a portion a from private investment and $(1-a)$ from consumption. Assume a unit of private investment produces a perpetuity paying i per period. Now if no project output is re-invested, the project is desirable if:

$$(1+\delta)/(1+r) \geq (1-a) + ai/r$$

The left-hand side is the present value of the project output. The right-hand side is the shadow price of capital, the present value of the consumption resources used in the project would have produced. This reduces to:

$$\delta \geq (1-a)r + ai + a(i-r)/r = w + a(i-r)/r$$

which substantially raises the rate of return the project must earn. For example, if $a = 0.75$, $i = 3$ per cent and $r = 1$ per cent (plausible risk-free numbers), then $w = 2.5$ per cent and $a(i–r)/r = 150$ per cent. The project must earn 152.5 per cent to be desirable – far above the weighted average rate – because it permanently reduces the private capital stock by 0.75 units, which reduces the present value of consumption by 0.75*0.03/0.01 = 2.25 units.

If some project output were re-invested, then that would increase the private capital stock, reduce the consumption cost of the project and decrease the required project return. If the private capital stock depreciates rather than produce a perpetuity, that would reduce the consumption cost from reducing private investment.[3]

In his influential and masterful survey of the social discount rate, Lind (1982a) recommends shadow pricing capital and discounting with the consumption rate. It is fair to say that very few cost–benefit guides or studies have followed the Lind recommendation because the informational requirements make it impractical. As Lind points out, estimating the shadow price of capital requires we have estimates of the marginal return on private capital, the consumption rate, the length of life of the typical private investment, the displacement and stimulative effects of public investment on private capital formation, the amount of re-investment of public and private project output.[4] The alternative of working out a threshold rate of return to use as a discount rate involves shadow pricing capital and so the information required is exactly the same.

But Lind (1982a) does not refer to the analysis of Sjaastad and Wisecarver (1977).[5] They independently show that re-investment of project output is a crucial determinant of the social discount rate. They also show that if society does not treat public–project depreciation as income subject to current consumption, but instead intend to save (and hence re-invest) all of that depreciation, the social discount rate is again the weighted average discount rate. Only the net return on capital is treated as current income to be consumed.

3 See Bradford (1975) for how the appropriate discount rate varies with the different assumptions about the re-investment of project output and life of the private capital stock. He finds that it can lie above i and below r.

4 Lind (1982a, (pp. 48-50, 77-78).

5 Lind does consider the re-investment issue on pp.48-50. Lind points out that Bradford (1975) assumes depreciation of private investment not re-invested but is treated as any other component of current income. Lind states it is 'reasonable to assume' (p.50) that private project depreciation is fully re-invested. Sjaastad and Wisecarver (1997) assume depreciation is saved in both the private and public sector. Recognising these different assumptions is needed to reconcile the equations in Bradford (1975) with those in Sjaastad and Wisecarver (1977).

Again consider a two–period project where 1 unit is invested and produces gross output $1+\delta$ one period later. Depreciation is 1, the investment fully depreciates in one period. If consumers intend to save depreciation, it shifts the savings supply curve to the right by 1 unit, having effects exactly symmetric with the extraction of 1 unit of capital in the first period. That is, it will reduce interest rates, increasing actual investment by a units and consumption by $(1\text{–}a)$ units. The intended re-investment of project depreciation increases actual re-investment by a. But that would, once public investment has fully depreciated, restore the total capital stock, and future income from that stock, to the paths that would have existed if the project had never been undertaken.[6] That is, the project only affects the private capital stock during the life of the project.

Whether the project increases efficiency involves comparing the increase in consumption in period 2 with the consumption forgone in period 1. The increase in consumption in period 2 is the net return from the project, δ, less the consumption forgone from the one-period reduction in private capital, ia, plus the extra consumption from the saving of project depreciation, $1\text{–}a$. That is, the project improves efficiency if:

$$(\delta - ia + 1 - a)/(1+r) \geq (1 - a) \text{ or } \delta \geq (1 - a)(1+r) + ia - (1-a) \text{ or}$$

$$\delta \geq (1 - a)r + ai.$$

That is, the present value of consumption is higher with the project when the project return exceeds the weighted average discount rate, w, or when the project has a positive net present value using a discount rate w (that is, $(1+\delta)/(1+r) \geq 1)$. Again the weighted average and shadow pricing approach are equivalent.

Further, the weighted average approach gives the same project rankings as the shadow pricing approach.[7]

If society recognises the difference between depreciation and net benefits from a public investment project and attempts to save all the benefits which represent depreciation, then the weighted average discount rate is the social discount rate. That people try to save depreciation may appear to be a strong assumption. Consumers of government provided goods are unlikely to be aware what part of

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6 Sjaastad and Wisecarver (1977, p. 522).

7 Project 1 has costs C_1 and benefits B_1. Project 2 has costs C_2 and benefits B_2. Project 1 is better under the shadow pricing approach if:

$$[B_1 - C_1 - iaC_1 + (1-a)C_1]/(1+r) - (1-a)C_1 > [B_2 - C_2 - iaC_2 + (1-a)C_2]/(1+r) - 1-a)C_2$$

where project depreciation is saved so the weighted average pricing approach applies. This reduces to: $B_1/(1+w) - C_1 > B_2/(1+w) - C_2$ which is the ranking when project flows are discounted with the weighted average discount rate, w.

project benefits represents net output and what part depreciation. On the other hand, if people do consume depreciation and run the capital stock down, it needs to be explained why they did not take this opportunity in the absence of the project. There are sound economic reasons for people to try to maintain the capital stock to keep consumption at desired levels.

It may be reasonable to assume re-investment of depreciation, especially with regulation. The government and firms can identify depreciation and reinvest it. For regulatory projects, firms often undertake the project investment, and recoup their costs from consumers. It is reasonable to assume that they can distinguish depreciation from net output and would save depreciation. If the government implicitly commits to provide the good or service beyond the life of the project, that requires reinvestment of public sector funds approximately equal to the depreciation of the project's capital.[8]

When depreciation is reinvested, the impact of the project on the economy lasts only for the duration of the project. That is appears more reasonable than the alternative — that a government capital project permanently alters the mix of consumption and investment in the economy.

Certainly the Sjaastad and Wisecarver approach is more practical. It only requires information the project flows, the investment and consumption rates, and the displacement of private investment. The shadow pricing approach needs all this plus the length of life of the typical private investment and the amount of re-investment of public and private project output. The rate of re-investment of project output would need to be calculated for each project, yet the information for the calculation is unlikely to be available.

In the absence of the necessary information, the most reasonable, convenient and practical approach is to assume society saves project depreciation and to discount with the weighted average discount rate, which makes best use of the information we do have. Thinking in terms of adjusting the discount rate is easier and more informative than the shadow pricing approach when we are ignorant.

If society does not save project depreciation, but consumes it, re-investment is less and the appropriate discount rate is greater than the weighted average rate, perhaps substantially greater. In this case, the project permanently reduces the private capital stock and the resulting reduction in consumption each period is a cost of the project. Further, the rate of discount must be higher the shorter is the life of the project, as

8 Jenkins (1981, p. 400).

short–term projects lead to capital consumption at an earlier date than long–lived projects.[9] Short term projects would face a high hurdle.

If people do save depreciation and also re-invest out of net project output (which increases α), that would reduce the social discount rate below w. Sjaastad and Wisecarver find this effect to be quantitatively small.[10]

It is common in the literature to extol the virtues of the shadow pricing approach, while simultaneously dismissing the weighted average approach as incorrect.[11] Sjaastad and Wisecarver show that if society saves depreciation, then the two approaches are identical. Whether that is a reasonable assumption can be criticised, but most critics of the weighted average approach do not. Instead they often make invalid criticisms.

To turn it around, the weighted average approach is only correct if society saves public project depreciation. When weighted average critics present counter-examples, they invariably assume (usually implicitly, and unwittingly) that depreciation is not saved. If the project output flows and corresponding changes in consumption when depreciation is saved are correctly specified, the weighted average and shadow pricing approach give the same answer. If critics specify project flows and changes in consumption inconsistent with depreciation being saved, then the weighted average approach does not give the correct answer.

For example, the United Stated Environmental Protection Agency (EPA) claims that the weighted average approach is 'acceptable for similarly timed cost and benefit flows' and

> it is technically incorrect and can produce net present value results substantially different from the correct result (where 'correct' is defined by the consumption rate of interest–shadow price of capital approach). The problem with the simple weighted average approach is that it seeks to accomplish two tasks using the social discount rate — pure time discounting and adjusting for the displacement of private investments that yield pre–tax social returns higher than the consumption rate of interest.[12]

The EPA claims the weighted average approach 'over discounts' long–lived projects, giving the wrong answer when benefits are far in the future, and that the problem is worse the farther in the future the benefits occur. The next section demonstrates that the claim is incorrect.

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9 Sjaastad and Wisecarver (1977, p. 523).

10 See Sjaastad and Wisecarver (1977, p. 527).

11 For example, Abelson (2000, p. 129), Bureau of Transport and Regional Economics (1999, p .71, footnote 30) and United States Environmental Protection Agency (2000, p. 4).

12 See United States Environmental Protection Agency (2000, p. 42) and Abelson (2000, p. 129) makes the same point.

D.4 Does the weighted average approach give the correct answer for long–term discounting?

As Sjaastad and Wisecarver prove, in a two-period project, when depreciation is saved then the shadow pricing and weighted average approach give the same answer. Any project can be expressed as a sequence of two period projects. If depreciation is saved at each step in the sequence, discounting the project flows with the weighted average discount rate gives the same answer as discounting the consumption flows with the consumption rate. The EPA does not allow for re-investment of depreciation in its calculations, or the consequences for consumption of not re-investing, and incorrectly specifies the relevant investments.

Take a project in which 1 unit is invested in period 0, and the only effect on consumption is to reduce it by $(1-a)$ units in period 0 and then increase it by $(1-a)(1+r)^n$ units in period n. That project would just break–even (we evaluate consumption changes with the consumption rate, r).

The project is equivalent to following sequence of two period projects, each with a gross rate of return equal to δ:

1. Invest 1 in period 0, consumption falls by $(1-a)$, private investment falls by a.

The investment gives gross output of $1+\delta$ in period 1. 1 is re-invested. The re-investment increases consumption by $(1-a)$ and private investment by a, restoring the private capital stock to its pre–project path. The project therefore gives $\delta - ia + (1-a) = z$ in (potential) extra consumption in period 1. Note that $z \geq (1-a)(1+r)$ if $\delta \geq ia+(1-a)r = w$.

Now invest $z/(1-a)$ in period 1. This will reduce period 1 consumption by z units, keeping consumption in period 1 unchanged. Private investment falls by $az/(1-a)$.

2. The potential consumption and fall in private investment fund the increased investment: $z+az/(1-a)=z/(1-a)$.
3. The investment allows $z^2/(1-a)$ (potential) extra consumption in period 2.
4. Invest $z^2/(1-a)^2$ in period 2. This keeps period 2 consumption unchanged and gives a potential increase in period 3 consumption of $z^3/(1-a)^2$.
5. Invest $z^j/(1-a)^j$ in period j $(j < n)$, which keeps consumption at the initial level in each period j.
6. When we reach period n, the final investment allows consumption of $z^n/(1-a)^{n-1}$. The project is efficient if the present value of the consumption

allowed in period n exceeds the consumption forgone in period 0: $z^n/(1–a)^{n-1}(1+r)^n \geq (1–a)$

7. If $\delta \geq w = ia + (1–a)r$ then $z = \delta – ia + (1–a) \geq (1–a)(1+r)$ and the condition holds. The return on government projects needs to at least w for an investment in period 0 to raise consumption by $(1–a)(1+r)^n$ in n periods and keep consumption constant in period $1, .., n–1$.

That is, if we discount the project flows with the weighted average discount rate w, it gives the same answer as discounting the changes in consumption with the consumption rate, r.

Take a simple example (based on the EPA example, but reduced to 3 periods so it can be presented in a short table). Assume the investment rate is 5 per cent, the consumption rate 3 per cent, $1 of public investment reduces private investment by $0.75 and consumption by $0.25. that is, $a = 0.75$. the weighted average discount rate is $w = 0.75*5 + 0.25*3 = 4.5$ per cent.

Now take a 3 period project where $1 is invested in period 0 which increases consumption in period 2 only. This is equivalent to a sequence of two 2-period projects. The investments earn a return of 4.5 per cent. Depreciation is saved for each project. The period by period flows are set out below in table D.1, 1 unit is invested in period 0, reducing consumption by 0.25 and private investment by 0.75 which produces output of 1.045 in period 1. Depreciation of 1 unit is saved, increasing the private capital stock by 0.75. The potential increase in consumption in period 1 is $\delta – ai + (1 – a) = 0.2575$. In period 1 a new investment of 1.03 is made, which reduces the capital stock by a*1.03 = 0.7725 and crowds out 0.25*1.03 = 0.2575 of consumption in period 1. the combined effect of the two projects is to keep period 1 consumption constant. The new investment produces output of 1.045*1.03 = 1.076 in period 2. Depreciation of 1.03 is saved, which restores the private capital stock to its initial level. Period 2 consumption rises by 1.076 – 1.03 – 0.05*0.7725 + 0.25 = 0.265

Using the shadow pricing approach, the project breaks even. The present value (discounted with the consumption rate 3 per cent) of the change in consumption from the project is $–0.25 + 0.265/(1.03)^2 = 0$.

Table D.1 **Project cost and benefit flows**

A three period example

Period	*New investment in project*	*Gross project output*	*Re-investment*	*Potential increase in consumption*	*Change in private investment from new investment*	*Change in consumption*
0	1	0			– a = – 0.75	–(1–a)=–0.25
1	1.03 = 0.2575/0.25	(1+*w*)=1.045	a*1 = 0.75	0.2575= (1+w)–1–ia+(1–a)	–0.7725 = –(1–a)*1.03	0
2	0	1.076= 1.03*1.045	0.7725	0.265= 1.076–1.03–0.05*0.7725 +0.25*1.03	0	0.265

The EPA suggests the weighted average approach incorrectly rejects this marginal project because if we discount the period 2 gross output at 4.5 per cent, the net present value is $1.076/(1.045)^2$ = \$0.986, less than the \$1 cost of the project. They fail to carefully specify the investments needed to produce the above consumption pattern or account for re-investment. They do not account for the fall in consumption in each project period from the project induced decline in the private capital stock. Further, they have no re-investment in the final project period, which would permanently lower the private capital stock, yet the EPA fails to account for the resulting fall in consumption which would continue after the project finishes. The weighted average approach, properly applied, calculates the present value of the gross project output less the present value of investments made. The net present value of the project flows (columns 2 and 3 in table D.1) is:

$1.076/(1.045)^2 + 1.045/1.045 - 1.03/1.045 - 1 = 0$, consistent with the shadow pricing approach.

A project can be expressed as a sequence of two period projects. Each of the two period projects would increase the present value of consumption only if it earned the weighted average rate of return.

The EPA suggest that a project which involved investing 1 unit in period 0 with a payoff of $(1.03)^2 = 1.0609$ in period 2, with no further investments, would be marginal. The project would reduce the private capital stock by 0.75 units and consumption by 0.25 in period 0. That would reduce consumption by 0.75*0.05 = 0.0375 units in period 1. In period 2 consumption would increase by: $1.0609 - 0.0375 - 1 + 0.25 = 0.2734$ if the public intend to save project depreciation (if it did not, then the capital stock would be permanently reduced, increasing the required project return substantially). The present value of the change in consumption is:

$$0.2734/1.03^2 - 0.0375/1.03 - 0.25 = -0.029$$

That is, the project return of 3 per cent is not enough to cover its costs. The EPA do not account for the fall in consumption in the intermediate periods from the fall in the private capital stock. Discounting with the consumption rate would only be correct if the government project did not crowd out private investment ($a = 0$), in which case the weighted average rate would be the consumption rate.

D.5 Discounting with the government bond rate

Many cost benefit guides, and practitioners, recommend the before–tax return on government bonds as the social discount rate. Sometimes using the government bond rate is justified as reflecting the cost of funds to the government.[13] But that does not account for the indirect cash flows when the government goes into debt. When we account for these flows, the weighted average discount rate gives the cost of borrowing. For example, if the parameters are as in the previous section, ($r = 3$ per cent, $i = 5$ per cent, $t = 40$ per cent, $a = 0.75$ and $w = 4.5$ per cent), when the government borrows $100 it:

- makes a $5 interest payment
- receives $2 in tax payments from those receiving the interest
- crowds out $75 of private investment, which reduces tax revenue by $75*0.05*0.4 = $1.50.

The total annual cost to the government of borrowing $100 is $5 – $2 + $1.50 = $4.50, or 4.5 per cent, the weighted average rate. Alternatively, extra tax revenue only comes from the newly stimulated savings, so is $25*0.05*0.4 = $0.50. So the total cost is $5 – 0.50 = $4.50, which is the weighted average rate times $100. The weighted average rate gives the cost of financing.

Really, whether the government bond rate is appropriate turns on how the cost benefit analysis accounts for the cost of risk, which is considered in the risk section.

[13] See, for example, Department of Treasury and Finance Tasmania (1996, p. 16).