A Linear interpolation method

There are two traditional ways of calculating the contribution of a factor to the growth of a multiplied quantity. However, these methods suffer from the disadvantage that the sum of the individual factor contributions do not add to the growth in the total. A 'linear interpolation contribution to growth' can be defined that overcomes this disadvantage.

Contributions to growth in a multiplied quantity (for example, $t = xyz$) can be calculated by finding the growth in **t** that would occur if only one factor were to be increased — ie $\Delta t_x = (\Delta x).y.z$. An alternative method is to calculate:

- the change in t were all factors to be increased
- the change were all factors excepting one to be increased
- then subtract one from the other so that:

$$\Delta t_x = (x+\Delta x).(y+\Delta y).(z+\Delta z) - x.(y+\Delta y).(z+\Delta z).$$

One difficulty with these approaches is that the contributions to growth of the various factors (Δt_x, Δt_y, Δt_z) will only add to the overall growth in t in the limiting case where the growth in t is infinitesimally small. This is because of the presence of cross-terms ($\Delta x.\Delta y$, $\Delta x.\Delta z$, $\Delta y.\Delta z$ and $\Delta x.\Delta y.\Delta z$, etc). Consider the example where $x_1 = 10$, $y_1 = 10$ and $z_1 = 100$. If each factor grew by the same proportion, say 10 per cent (so that $\Delta x = 1$, $\Delta y = 1$, $\Delta z = 10$), then the growth in the total (Δt) would be 3310 or 33.1 per cent. In contrast, using the first method above would give $\Delta t = (\Delta x).y.z+(\Delta y).x.z+(\Delta z).x.z = 3000$.

A more sophisticated approach is to define a 'linear interpolation contribution to growth' (*CLI*) for each factor, with the property that the sum of individual linear interpolation contributions is identical to the total growth in *t*. This linear interpolation contribution is defined as the integral of a factor's partial contributions to growth[1] calculated at every point along a straight line interval of growth in *t*.

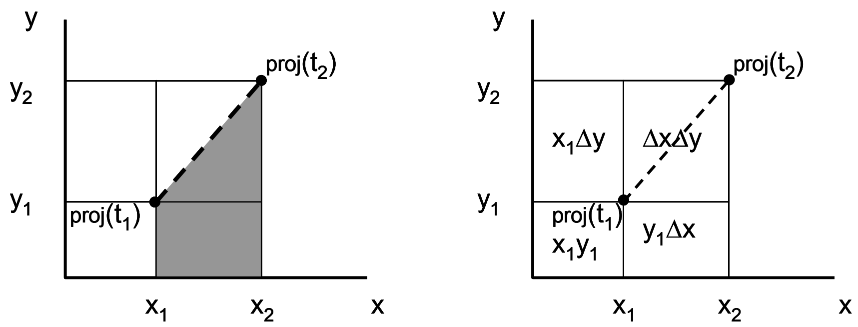
[1] A partial change is the change in the total that would occur were one factor to increase by a very small amount with all other factors remaining unchanged.

A particular *CLI* can be approximated by:

- dividing an actual growth in t between two points into a very large number of small steps along a straight line
- finding a particular factor's partial contribution after each of these small steps
- summing the partial contributions.

In the two factor case, the line along which the integral is taken is the line formed by the projection of the $t = xy$ line (which is a line in three dimensions) on the *xy* plane. The first panel in figure A.1 below shows a graphical example.

Figure A.1 **Linear interpolation contribution of the growth in *x* to the growth in *t = xy***



Contributions to growth over a curved growth path can also be approximated by dividing the path into a number of linear segments, then summing *CLI*'s calculated for each segment.

Deriving the linear interpolation contribution algebraically

The linear interpolation contribution can be calculated using areas as shown in the second panel of figure A.1.[2] For example, in the two factor case the calculation is as follows:

$(t+\Delta t) = (x+\Delta x)(y+\Delta y)$

$(t+\Delta t) = xy + y.\Delta x + x.\Delta y + \Delta x.\Delta y$

$\Delta t = y.\Delta x + x.\Delta y + \Delta x.\Delta y$

The linear interpolation contribution of x is then $y.\Delta x + ½.\Delta x.\Delta y$

2 The linear interpolation was originally derived using limits (see PC 2005, Technical Paper 6)

For the three factor case the calculation is $(t+\Delta t) = (x+\Delta x).(y+\Delta y).(z+\Delta z)$. So, after removing $t = xyz$:

$$\Delta t = yz.\Delta x + xz.\Delta y + xy.\Delta z + z.\Delta x.\Delta y + y.\Delta x.\Delta z + x.\Delta y.\Delta z + \Delta x.\Delta y.\Delta z$$

Now, since the three linear interpolation contributions share mixed partial product terms equally, this implies that the linear interpolation contribution of factor *x* to the growth in t is $\Delta t_{(cli)x} = yz.\Delta x + ½.z.\Delta x.\Delta y + ½.y.\Delta x.\Delta z + ⅓.\Delta x.\Delta y.\Delta z$.

(Below we prove that this formula is the correct one for three factors without relying on the assertion that the mixed factor terms are shared equally.)

To continue with the above example (where $x_1 = 10$, $y_1 = 10$, $z_1 = 100$, $\Delta x = 1$, $\Delta y = 1$ and $\Delta z = 10$), then the linear interpolation contributions of each factor will be $\Delta t_{(cli)x} = \Delta t_{(cli)y} = \Delta t_{(cli)z} = 1103.3333$, which sum to the total growth in *t*.

Proving the linear interpolation formula for three factors holds using integration

The formula for the linear interpolation contribution can also be derived using area, volume and higher dimensional integrals. This is useful because it is a reasonably simple way of showing the *CLI* formula holds when there are more than two factors.

With two factors, the function $t = xy$ measures the size of an area. The change in the amount of this area (Δt) is given by $x_2y_2 - x_1y_1$ (a larger rectangle less a smaller rectangle). So the *CLI* of *x* is the area between the projection of the $t = xy$ line on the *xy* plane and the *x* axis (this is the shaded area shown in the first panel of figure A.1 above), while the CLI of y is the area between the projection of the $t = xy$ line on the *xy* plane and the *y* axis.

With three factors the function $t = xyz$ measures the size of a volume. Here the Δt volume is equivalent to a larger rectangular box with a smaller rectangular box removed ($x_2y_2z_2 - x_1y_1z_1$). The linear interpolation contributions of changes in *x*, *y* and *z* are then three volumes that add to the this total volume. The *CLI* of *x* is the volume found by integrating in the *z*, then *y* and then *x* directions[3] or in the *y* then *z* then *x* directions. Similarly, The *CLI* of *y* is the volume found by integrating in the *y* direction last and the *CLI* of *z* is the volume found by integrating in the *z* direction last.

[3] That is integrating in the *z* direction between the $z = 0$ and the $z = f_1(x)$ planes, then integrating in the *y* direction the between $y = 0$ and $y = f_2(x)$ lines and then integrating in the *x* direction between the x_1 and x_2 points.

It is also possible to find the equivalent of the *CLI* except now for a curve rather than a straight line. That is the integral of the partial changes in t for changes in x over a curved line segment divided into an infinite number of increments (i_n). This can be designated by:

$$\int_i \frac{\partial t}{\partial x}$$

The curved line formed from the various points (x_1,y_1,z_1), (x_2,y_2,z_2), (x_3,y_3,z_3) etc. can be used to calculate the various CLI integrals (where t_n measures the volume of a box that has the points (0,0,0) and (x_n,y_n,z_n) as two of its vertices and has as three of its sides parts of the xy, xz and yz planes). The projection of this curved line on the xz plane is the curved line $z = f_1(x)$ and its projection on the xy plane is the curved line $y = f_2(x)$.

So with three factors: $\int_i \frac{\partial t}{\partial x} = \int_{x1}^{x2}\int_0^{f2(x)}\int_0^{f1(x)} 1 \, dz \, dy \, dx = \int_{x1}^{x2} f_1(x).f_2(x) \, dx$

In the linear case[4] we substitute $z = f_1(x)$ and $y = f_2(x)$ with $z = \alpha_1 x$ and $y = \alpha_2 x$, so:

$$\Delta t_{(cli)x} = \int_i \frac{\partial t}{\partial x} \text{(linear)} = \int_{x1}^{x2} \alpha_1\alpha_2 x^2 \, dx \quad \therefore \Delta t_{(cli)x} = ⅓\alpha_1\alpha_2(x_2^3 - x_1^3)$$

It can be shown that this is equivalent to $yz.\Delta x + ½.z.\Delta x.\Delta y + ½.y.\Delta x.\Delta z + ⅓.\Delta x.\Delta y.\Delta z$ where:

$z = \alpha_1 x_1$, $y = \alpha_2 x_1$, $\Delta x = x_2 - x_1$, $\Delta y = \alpha_2(x_2 - x_1)$ and $\Delta z = \alpha_1(x_2 - x_1)$

After substituting it can be seen that $yz.\Delta x + ½.z.\Delta x.\Delta y + ½.y.\Delta x.\Delta z + ⅓.\Delta x.\Delta y.\Delta z$

$$= \alpha_1\alpha_2 x_1^2(x_2 - x_1) + ½\alpha_1\alpha_2 x_1(x_2 - x_1)^2 + ½\alpha_1\alpha_2 x_1(x_2 - x_1)^2 + ⅓\alpha_1\alpha_2(x_2 - x_1)^3$$

$$= \alpha_1\alpha_2(x_1^2 x_2 - x_1^3) + \alpha_1\alpha_2 x_1(x_2^2 - 2x_1x_2 + x_1^2) + ⅓\alpha_1\alpha_2(x_2^3 - 3x_1x_2^2 + 3x_1^2x_2 - x_1^3)$$

$$= \alpha_1\alpha_2(x_1^2x_2 - x_1^3 + x_1x_2^2 - 2x_1^2x_2 + x_1^3 + ⅓x_2^3 - x_1x_2^2 + x_1^2x_2 - ⅓x_1^3)$$

$$= \alpha_1\alpha_2(x_1^3 - x_1^3 + x_1^2x_2 + x_1^2x_2 - 2x_1^2x_2 + x_1x_2^2 - x_1x_2^2 + ⅓x_2^3 - ⅓x_1^3)$$

$$= ⅓\alpha_1\alpha_2(x_2^3 - x_1^3)$$

[4] The alphas are simply constants. They can be calculated using values of x, y and z. For example $\alpha_1 = (z_2-z_1)/(x_2-x_1)$.

Summary

The linear interpolation contribution to growth (for a total that is the product of a number of factors) is the *integral of the partial changes in the total ascribed to a particular factor*, evaluated over a linear segment of growth in the total.

In order to derive the formula for the linear interpolation contribution it is useful to think of the two factor case in terms of areas and the three factor case in terms of volumes.

In the two factor case, the linear interpolation contribution change in **t** for a change in a particular factor is the area between the axis of the factor in question and the projection of the line t = *xy* on the *xy* plane evaluated over an interval.

$$\Delta t_{(cli)x} = \int_i \frac{\partial t}{\partial x}\,(\text{linear}) = \int_{x_1}^{x_2}\int_0^{\alpha_1 x} 1\ dy\ dx = \tfrac{1}{2}\alpha_1(x_2^2 - x_1^2) = y.\Delta x + \tfrac{1}{2}.\Delta x.\Delta y$$

And the two linear interpolation contributions add to the total change:

$$\Delta t_{(linear)} = \Delta t_{(cli)x} + \Delta t_{(cli)y}$$

In the three factor case, the linear interpolation contribution change in *t* for a change in **x** is equivalent to the volume found using the triple integral:

$$\Delta t_{(cli)x} = \int_i \frac{\partial t}{\partial x}\,(\text{linear}) = \int_{x_1}^{x_2}\int_0^{\alpha_2 x}\int_0^{\alpha_1 x} 1\ dz\ dy\ dx = \tfrac{1}{3}\alpha_1\alpha_2(x_2^3 - x_1^3)$$

$$= yz.\Delta x + \tfrac{1}{2}.z.\Delta x.\Delta y + \tfrac{1}{2}.y.\Delta x.\Delta z + \tfrac{1}{3}.\Delta x.\Delta y.\Delta z$$

And the three linear interpolation contributions add to the total change:

$$\Delta t_{(linear)} = \Delta t_{(cli)x} + \Delta t_{(cli)y} + \Delta t_{(cli)z}$$