C Theoretical framework

The mathematical formulation of the Commission's approach is outlined in this appendix.

C.1 Modelling hospital mortality with negative binomial regressions

Modelling in-hospital mortality using hospital-level data requires a statistical model that takes into account the fact that the number of in-hospital mortalities is a non-negative integer.

The Poisson model is the simplest count model, and describes the number of occurrences of an event, m_i (such as the number of mortalities occurring in hospital i) as:

$$m_i = \frac{e^{-\lambda}\lambda^{m_i}}{m_i!} \tag{1}$$

where the mean and variance of m are both equal to λ, where:

$$\lambda = e^{\mathbf{x}_i\boldsymbol{\beta}} \tag{2}$$

where $\mathbf{x}_i$ is a vector of explanatory variables of hospital i.

This model, however, holds only if the mean and variance are equal. This is not the case with hospital mortality data, which include a large number of hospitals with relatively few deaths and a smaller number of large and very large hospitals with comparatively more deaths. This is a case of over-dispersion, where the variance is significantly greater than the mean.

If the variance of the dependent variable is greater than the mean, λ can be specified as:

$$\lambda = e^{\mathbf{x}_i\boldsymbol{\beta}+\varepsilon_i} \tag{3}$$

where error term ε_i has a gamma distribution. This in turn leads to a negative binomial distribution for the number of deaths with mean λ and variance:

$$\sigma^2 = \lambda + \alpha\lambda^2 \quad (4)$$

where α represents the level of over-dispersion.

The test of whether a Poisson or negative binomial model would be appropriate would depend upon the statistical significance of α. If α is significant, as it is in this study, the negative binomial would be the appropriate model. Otherwise, the Poisson model would be appropriate (Cameron and Trivedi 2005; Winkleman and Boes 2006).

Under such a model, it is assumed that:

- there is a mortality rate — the rate at which deaths occur
- the mortality rate can be multiplied by an 'exposure' to determine the expected number of deaths. In this case, the exposure is the number of casemix-adjusted separations
- over very small exposures, the probability of observing more than one death is small compared to the size of the exposure (Cameron and Trivedi 2005; Kennedy 2003; Winkleman and Boes 2006).

C.2 Approximating confidence intervals for hospital-standardised mortality ratios

In presenting hospital-standardised mortality ratios (HSMRs) for individual hospitals, as in the 'caterpillar' plots in chapter 4, it is important to acknowledge the uncertainty that is inherent in estimated values. As noted by Ben-Tovim, Woodman, Harrison et al. (2009), the conventional method of incorporating a measure of uncertainty is by calculating the confidence intervals around each HSMR. This is usually done using 95 percent confidence limits (Ben-Tovim, Woodman, Harrison et al. 2009; CIHI 2007; Lakhani, Olearnik and Eayres 2005). It is expected that the HSMR will be within this range 95 per cent of the time on repeat testing of a population. The size of the confidence interval indicates the precision of the HSMR.

The confidence intervals shown in the caterpillar plots are calculated in the same manner as CIHI (2007) and Ben-Tovim, Woodman, Harrison et al. (2009), using what is referred to as Byar's approximation. Given that:

$$HSMR_i = \frac{deaths_i}{E(deaths_i \mid \mathbf{X}_i)} \text{ x } 100 \qquad (5)$$

where $\mathbf{X}_i$ represents a vector of patient and hospital characteristics, the lower confidence limit for hospital *i* is given by the equation:

$$HSMR_{iLL} = \frac{deaths_i}{E(deaths_i \mid \mathbf{X}_i)} \text{ x } \left(1 - \frac{1}{9(deaths_i)} - \frac{1.96}{3\sqrt{deaths_i}}\right)^3 \text{ x } 100 \qquad (6)$$

Similarly, the upper confidence limit is given by:

$$HSMR_{iUL} = \frac{(deaths_i + 1)}{E(deaths_i \mid \mathbf{X}_i)} \text{ x } \left(1 - \frac{1}{9(deaths_i + 1)} - \frac{1.96}{3\sqrt{(deaths_i + 1)}}\right)^3 \text{ x } 100 \qquad (7)$$

This is explained in more detail in the Compendium of Clinical and Health Indicators user Guide (Lakhani, Olearnik and Eayres 2005).

C.3 Estimating hospital efficiency

Estimating distance functions

The distance function is the stochastic frontier analysis analogue of multi-output multi-input production. The function can be specified as an:

- output distance function — which measures the maximum amount by which outputs can be expanded and still be producible with the given set of inputs
- input distance function — which measures the maximum amount by which inputs can be reduced and still remain feasible for the outputs they produce (Kumbhakar and Lovell 2000).

The output distance function is more appropriate for hospitals that can influence their level of outputs, such as private hospitals. The input distance function is more appropriate for measuring the technical efficiency for hospitals that find it difficult to reduce their output but which are able to alter their use of inputs, such as public hospitals. These two approaches also allow the relationship between hospital quality and changes to input use and hospital outputs to be explored.

Output distance function

For the output distance function, the production technology of the hospital is defined with the output set *P(x)* which represents the set of all output vectors $y \in R_+^M$ that can be produced using the input vector $x \in R_+^K$. An output distance function is defined by how much the output vector can be proportionally expanded by amount θ with the input vector held fixed (Coelli and Perelman 1999; Lovell et al. 1994). The output distance function may be defined on the output set as:

$$D_O(\mathbf{x},\mathbf{y}) = \min\left\{\theta : (\mathbf{y}/\theta) \in P(\mathbf{x})\right\} \tag{8}$$

The output distance function will take a value of one or less if the output vector **y** is an element of the feasible output set. If **y** is on the outer boundary of the input set, the distance function will take a value of one.

The translog output distance function for hospital *i* is given as:

$$\ln D_{Oi} = TL(\mathbf{x}_i, \mathbf{y}_i, q_i, \mathbf{z}_i; \boldsymbol{\beta}) \tag{9}$$

The homogeneity constraints are that outputs are homogenous to degree one in outputs, given by (Coelli et al. 2005; PC 2009). These constraints can be met by normalising equation (9) by the K*th* output:

$$\ln\left(\frac{D_{Oi}}{y_{iK}}\right) = TL(\mathbf{x}_i, \mathbf{y}_i^*, q_i, \mathbf{z}_i; \boldsymbol{\beta}) \tag{10}$$

where **y*** is the vector of normalised outputs. Equation (10) can be re-arranged with a random error term to give a variable returns to scale output distance function:

$$-\ln y_{iK} = TL(\mathbf{x}_i, \mathbf{y}_i^*, \mathbf{z}_i; \boldsymbol{\beta}) - \ln D_{Oi} + v_i \tag{11}$$

where *TL*(.) refers to the transcendental logarithmic (translog) function. In the output distance function, hospital quality q_i is interacted with the $\boldsymbol{y}_i$ vector to test whether there are economies of scope between hospital activity and mortality rates. Again, the dependent variable is multiplied by –1 to ensure that the coefficients on the right-hand side reverse their signs.

Input distance function

The production technology of the hospital is defined with the input set *L*(***y***) which represents the set of all input vectors $x \in R_+^K$ that can produce the output vectors

$y \in R_+^M$. An input distance function is defined by how much the input vector can be proportionally contracted by amount ρ with the output vector held fixed (Coelli and Perelman 1999; Lovell et al. 1994). The input distance function may be defined on the input set as:

$$D_I(\mathbf{x},\mathbf{y}) = \max\{\rho : (\mathbf{x}/\rho) \in L(\mathbf{y})\} \tag{12}$$

The input distance function will take a value of one or more if the input vector **x** is an element of the feasible input set. If **x** is on the inner boundary of the input set, the input distance function will take a value of one.

The translog of the input distance function is given as:

$$\ln D_{Ii} = TL(\mathbf{y}_i, \mathbf{x}_i, q_i, \mathbf{z}_i; \boldsymbol{\beta}) \tag{13}$$

The input distance function must be homogeneous of degree one in inputs (Coelli and Perelman 1999). These conditions can be met by normalising the inputs by the M*th* input:

$$\ln\left(\frac{D_{Ii}}{x_{iM}}\right) = TL(\mathbf{y}_i, \mathbf{x}_i^*, q_i, \mathbf{z}_i; \boldsymbol{\beta}) \tag{14}$$

where **x*** is the vector of normalised inputs. Rearranging the left-hand side variables and adding a random error term v_i, we obtain the equation to estimate variable returns to scale:

$$-\ln x_{iM} = TL(\mathbf{y}_i, \mathbf{x}_i^*, q_i, \mathbf{z}_i; \boldsymbol{\beta}) - \ln D_{Ii} + v_i \tag{15}$$

where D_{Ii} is equal to the input-oriented distance and technical efficiency. In the translog functional form, the hospital quality q_i is interacted with vector $\mathbf{x}_i$, to test the extent to which there are economies of scope between input use and (standardised) mortality rates. In chapter 5, the input-oriented technical efficiency is inverted (divided into 1) for ease of interpretation.

Estimating cost functions

The estimation of hospital cost efficiency begins with a model of hospital costs which takes the general form of:

$$c_i = f(\mathbf{w}, \mathbf{y}_i, q_i, \mathbf{z}_i)\exp(v_i - ce_i) \tag{16}$$

where hospital i's total cost c_i is a function of $\mathbf{w}$ (the vector of input prices), $\mathbf{y}_i$ (the vector of outputs), q_i (the hospital-standardised mortality ratio), $\mathbf{z}_i$ (a vector of factors outside the control of hospitals), ce_i (the measure of cost efficiency) and v_i (the random error term).

The translog variable returns to scale equation takes the form:

$$\begin{aligned}\ln c_i = {} & \alpha_0 + \sum_{m=1}^{M} \alpha_m \ln w_m + \sum_{k=1}^{K} \beta_k \ln y_{ki} \\ & + 0.5 \sum_{m=1}^{M} \sum_{n=1}^{M} \alpha_{mn} \ln w_m \ln w_n + 0.5 \sum_{k=1}^{K} \sum_{l=1}^{K} \beta_{kl} \ln y_{ki} \ln y_{li} \\ & + \sum_{k=1}^{K} \sum_{m=1}^{M} \gamma_{km} \ln w_m \ln y_{ki} + \delta\, q_i + \sum_{k=1}^{K} \varepsilon_k\, q_i \ln y_{ki} \\ & + \sum_{p=1}^{P} \zeta_p z_{pi} - ce_i + v_i\end{aligned} \tag{17}$$

for M number of inputs, and K number of outputs. Equation (17) is not a fully specified translog function. This equation is a restricted translog function, since quality variables are assumed to interact with outputs only. Since the vector $\mathbf{z}$ represents control variables, these are assumed not to interact with other variables.

As the cost frontier needs to be linearly homogenous in input prices, c_i and input prices $w_1, \ldots, w_{M-1}$ are normalised by the input price of the M*th* factor w_M, so that:

$$\begin{aligned}\ln\left(\frac{c_i}{w_M}\right) = {} & \alpha_0 + \sum_{m=1}^{M-1} \alpha_m \ln\left(\frac{w_m}{w_M}\right) + \sum_{k=1}^{K} \beta_k \ln y_{ki} \\ & + 0.5 \sum_{m=1}^{M-1} \sum_{n=1}^{M-1} \alpha_{mn} \ln\left(\frac{w_m}{w_M}\right) \ln\left(\frac{w_n}{w_M}\right) + 0.5 \sum_{k=1}^{K} \sum_{l=1}^{K} \beta_{kl} \ln y_{ki} \ln y_{li} \\ & + \sum_{k=1}^{K} \sum_{m=1}^{M-1} \delta_{km} \ln y_{ki} \ln\left(\frac{w_m}{w_M}\right) + \delta\, q_i + \sum_{k=1}^{K} \varepsilon_k\, q_i \ln y_{ki} \\ & + \sum_{p=1}^{P} \zeta_p z_{pi} - ce_i + v_i\end{aligned} \tag{18}$$

It is worth noting that $\mathbf{w}$ is assumed not to vary across individual hospitals, but reflects the market price of inputs faced by each private and public hospital sector in each jurisdiction. A more compact notation for the translog is:

$$\ln\left(\frac{c_i}{w_M}\right) = TL(\mathbf{w}^*, \mathbf{y}_i, q_i, \mathbf{z}_i; \boldsymbol{\beta}) - ce_i + v_i \tag{19}$$

where $TL(.)$ indicates that the function has a translog form and $\mathbf{w}^*$ indicates that the input price variables are normalised.

A challenge for measuring public costs is the absence of reliable estimates of capital costs. Public hospital accounting systems rarely account for depreciation and the opportunity cost of capital given the historical pattern of hospital funding. In the absence of adequate capital costs and capital prices, a short-run cost function is used, which is given as:

$$\ln\left(\frac{c_i}{w_M}\right) = TL(\mathbf{w}^*, \mathbf{y}_i, q_i, k_i, \mathbf{z}_i; \boldsymbol{\beta}) - ce_i + v_i \qquad (20)$$

where k_i is the number of hospital beds and is a proxy for the capital stock in the hospital. In chapter 6, the cost efficiency is inverted (divided into 1) to assist comparisons with the technical efficiency scores.

Testing for statistical differences between hospitals

Whereas the distance and cost functions describe the determinants of what constitutes hospital best practice, many authors have long sought to identify the factors that could possibly explain their reported efficiency. In the case of this study, this includes identifying if there is a statistical difference between different hospital ownership groups.

There are two commonly used ways in which additional variables have been be used to explain variations in efficiency. One approach is to regress the explanatory variables on the efficiency scores themselves. This is possible with stochastic frontier analysis of the Aigner–Lovell–Schmidt type because unlike traditional ordinary least squares, the residuals are not orthogonal to the regressors. This approach can be presented as:

$$u_i = f(\mathbf{Z}_i) + \varepsilon_i \qquad (21)$$

This approach is frequently used in techniques such as data envelopment analysis, where the data envelopment analysis (DEA) scores (u_i) are regressed on a number of other variables (z_i) to derive *conditional* DEA scores.

A problem with this two-stage approach is a lack of consistency in the assumptions about the distribution of the efficiency scores. In the first stage, the scores were assumed to be independently and identically distributed in order to estimate their values. However, in the second stage they were assumed to be a function of a number of firm-specific factors and are therefore not identically distributed (Coelli et. al 2005).

In the case of stochastic frontier analysis, it is possible to estimate all of the parameters, including those that might affect the inefficiency score and the random error term, in the same likelihood function.

For a simple production function, the combined regression would be:

$$\ln y_i = \beta_0 + \sum_{m=1}^{M} \beta_{mi} x_{mi} - u_i + v_i \tag{22}$$

$$\mu_i^u = \delta_0 + \sum_{j=1}^{J} \delta_{ji} z_{ji} + \xi_i \tag{23}$$

where μ_i^u are the conditional means of u for hospital i, j is the covariate subscript for hospital i, and ξ_i is the independently and identically distributed error.

Other issues

As is common practice, all terms are specified in natural logarithms, except shares and binary variables, so that the measures represent proportional values rather than absolute levels. All variables to be logged that have a natural value of zero are assigned a value of zero in the transformed dataset. Battese's (1996) method is used to correct for the bias this approach introduces. All logarithmic variables are mean corrected.

The first line of equation (18), which comprises first-order variables only, represents the standard Cobb-Douglas form. The inclusion of the higher-order squared terms in the second and third lines represents the complete translog function (Nguyen and Coelli 2009; Siciliani 2006).

Reporting efficiencies

The preceding equations produce a variety of ways to view and measure hospital efficiency. The following table (table C.1) summarises some of the more important efficiency dimensions used in this study and in which chapters they are reported.

Table C.1 **Summary of efficiency scores used in this study**

Description	*Summary of equation*	*Chapter reported in*
Output-oriented technical efficiency	D_{Oi} from equation (11)	5
Input-oriented technical efficiency	D_{Ii} from equation (15)	5
Average technical efficiency	$\left(D_{Ii}+D_{Oi}\right)/2$	5
Cost efficiency	ce_i from equation (19)	6

Interpreting the distance function coefficients

How are the coefficients of the estimable equations (13) and (15) to be interpreted? These are used to determine the parameters of the distance functions described in equations (9) and (13).

Following Perelman and Santin (2005), the elasticities of outputs and inputs with respect to distance (*D*) are given as:

$$r_{D,x_k}=\frac{\partial D}{\partial x_k}=\frac{\partial \ln D(x,y)}{\partial \ln x_k}\frac{D(x,y)}{x_k} \qquad (24)$$

$$r_{D,y_m}=\frac{\partial D}{\partial y_m}=\frac{\partial \ln D(x,y)}{\partial \ln y_m}\frac{D(x,y)}{y_m} \qquad (25)$$

A positive coefficient in the output distance function is associated with an increase in efficiency (distance), and a negative value is associated with a decrease in efficiency. In this study, since the dependent variable y_m has been pre-multiplied by –1, the interpretation of the coefficients is reversed.

At the same time, a positive coefficient in the input distance function is associated with a decrease in efficiency and a negative value is associated with an increase in efficiency. Again, since the dependent variable x_k has been pre-multiplied by –1, the interpretation of the coefficients is reversed.

The effect of a small change of an input on an output can be assessed in terms of the partial derivatives:

$$s_{y_m,x_k}=\frac{\partial y_m}{\partial x_k}=\frac{r_{D,x_k}}{r_{D,y_m}} \qquad (26)$$

The effect of a small change of an output on another output (through the marginal rate of transformation of outputs) is given by:

$$s_{y_m,y_n} = \frac{\partial y_n}{\partial y_m} = -\frac{r_{D,y_m}}{r_{D,y_n}} \qquad (27)$$

And the effect of a small change of an input on another output (through the marginal rate of substitution) is given by:

$$s_{x_k,x_j} = \frac{\partial x_k}{\partial x_j} = -\frac{r_{D,x_j}}{r_{D,x_k}} \qquad (28)$$

C.4 Evans and Walker indexes

The Evans and Walker information indexes are measures of the relative complexity of work undertaken by hospitals. They are based on work undertaken by Thiel (1967) in the field of information theory. Evans and Walker (1972) postulated a relationship between the complexity of work undertaken by a hospital and the information the hospital learns from undertaking that work. By establishing a link between complexity and information gain, the authors were able to adapt information indexes as proxies for hospital complexity.

In general, the amount of information a hospital learns from an admission is inversely related to the likelihood of that case occurring within the system and the likelihood of that hospital treating that particular case. If an event is almost certain to take place, such as a routine case from which the hospitals learns little, the hospital attracts a relatively low index of information gain (Butler 1988). In contrast, more complex (and presumably rarer) cases attract more information gain.

Evans and Walker offer two indexes. They differ in terms of the assumptions about the prior knowledge of probabilities. The first assumes there is no prior knowledge of the distribution of cases among hospitals. This is a measure of the complexity of a hospital's caseload (Evans and Walker 1972). The index X_i^1 is given as:

$$X_i^1 = \sum_j \bar{H}_j^1 p_{ij} \qquad (29)$$

which is a weighted average of the standardised complexity indexes $\bar{H}_j^1$ of each Australian refined diagnostic-related group (AR-DRG), where the weights p_{ij} are the share of the i*th*'s hospital's cases being classified as the j*th* AR-DRG.

To derive $\bar{H}_j^1$, the index of complexity for the j*th* AR-DRG is used:

$$H_j^i = \sum_i q_{ij} \ln\left(\frac{I}{q_{ij}}\right) \tag{30}$$

Equation (30) describes the information gain arising from the probability of the j*th* AR-DRG being treated by the i*th* hospital. Since I is the number of hospitals, the lower the probability q_{ij}, the larger will be their combined natural logarithm, and the information gain. Pre-multiplying gives the probability of that information gain occurring.

H_j^1 is standardised to ensure that the index has a mean of one:

$$\bar{H}_j^1 = \frac{H_j^1}{\sum_j H_j^1 q_j} \tag{31}$$

This second measure of a hospital's relative complexity takes into account the relative differences in hospital size. In this index, it is assumed that the prior probability of a case occurring is equal to the actual proportion of all cases in the system treated by the hospital. This means that the larger the hospital, the higher will be the probability that it will treat a case entering the system (Butler 1995). Rather than being a total measure of complexity as in the first index, the second measure is divided by the expected complexity faced by a hospital .

The second Evans and Walker index X_i^2 resembles the first, insofar that it is equal to the weighted average of standardised complexity cases $\bar{H}_j^2$:

$$X_i^2 = \sum_j \bar{H}_j^2 p_{ij} \tag{32}$$

However, the corresponding measure of information gain differs in that it is now influenced by the probability p_i that a case will go to the i*th* hospital:

$$H_j^2 = \sum_i q_{ij} \ln\left(\frac{q_{ij}}{p_i}\right) \tag{33}$$

As with the first index, equation (33) is standardised to ensure that the index has a mean of one:

$$\bar{H}_j^2 = \frac{H_j^2}{\sum_j H_j^2 q_j} \tag{34}$$

C.5 Hospital beds

Differences in the definitions used by public and private hospital sectors on the reported number of beds have the potential to distort the measured relative efficiencies of public and private hospitals. Public hospitals report the number of beds that are staffed (AIHW 2009), whereas private hospitals are asked to report the number of beds that are available (ABS 2008a). Since the number of beds that are physically available is at least as great and usually more than the number of beds that are staffed at any point, on the basis of publicly available data, there is likely to be inconsistent reporting of public and private hospital beds.

There do not appear to be any data at the national level on the number of physically available beds in public hospitals or the number of staffed beds in private hospitals. Instead, the Commission sought to estimate the number of private hospital beds that are staffed, to develop a measure consistent with the method of counting public hospital beds

Following Anson-Dwamena and Studer (2009), the average number of patients on any given day in a private hospital is given by:

$$Patients\ per\ day = \frac{Patient\ days}{365} \qquad (35)$$

Since each patient requires a bed, equation (35) gives the average number of beds the hospital needs staffed per day. However, since the demand for beds varies day-to-day, hospitals need additional beds to be staffed.

Assuming that the variation in the demand for beds follows a normal or Poisson distribution, the standard deviation is given as:

$$SD = \sqrt{\frac{Patient\ days}{365}} \qquad (36)$$

If the hospital administrators wish to ensure that beds are available 95 per cent of the time in a hospital, then the additional number of beds required will be 1.96 times the standard deviation of beds, so that in total:

$$No.\ of\ staffed\ beds = \frac{Patient\ days}{365} + 1.96\sqrt{\frac{Patient\ days}{365}} \qquad (37)$$

This rather simplistic approach does not take into account the effect of sameday separations and the different mix of beds in hospitals (ICU, acute, sub-acute and non-acute, for example). It does, however, reasonably predict the number of staff beds in public hospitals (table C.2).

Table C.2 **Estimated number of staffed beds in public and private hospitals**

	Estimated number of staffed beds	*Actual number of staffed beds*	*Actual number of physically available beds*	*Relative difference between (3) and (1)*
	(1)	(2)	(3)	(4)
	No. of beds	No. of beds	No. of beds	Per cent
Public hospitals	122.89	122.61	..	na
Private hospitals	118.14	..	123.64	4.65
Public contract hospitals	179.98	..	186.64	3.70

.. Not available. **na** Not applicable.

Source: Productivity Commission calculations based on AIHW and ABS data.

It is not clear to the Commission how public contract hospitals report the number of beds, as to whether these are ounts of the staffed or physically available beds. The Commission has assumed that, across the group of public contract hospitals as a whole, they are reporting physically available beds because they are privately owned or managed.

It is apparent that the predicted number of staffed beds in public hospitals is close to the reported number of staffed beds (table C.2). To ensure comparability in the measure of beds between the three groups of hospitals, the Commission used estimates of the number of staffed beds for private and public contract hospitals. On average, the number of physically available beds is between 4 and 5 per cent greater than the predicted number of staffed beds for private hospitals, and between 3 and 4 per cent greater for public contract hospitals.