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The Net Social Revenue Approach to Solving Computable General Equilibrium Models

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The Net Social Revenue Approach to Solving Computable General Equilibrium Models

Abstract

In this paper, we set out a new way to solve Computable General Equilibrium (CGE) models that contain inequality constraints and the associated complementarity (slackness) conditions. The approach is an extension of the Net Social Revenue formulation of Spatial and Temporal Pricing and Allocation models by Takayama and Judge. Our nonlinear programming framework to developing and solving CGE models expands the scope for the types of models that might be applied to evaluating economic policy, potentially improving policy relevant analysis. The nonlinear programming approach also increases the number of computer software packages available for both building and solving CGE models, including open source packages. The properties of the model (zero objective function value and symmetry of Lagrangean and model variables at the optimal solution) are also useful for developing and calibrating a model and verifying that it is correctly specified. Make a mistake and these do not hold. The framework presented in this paper is not appropriate for every exercise. The cost of the additional effort and time required to build a complete, integrated model might exceed the benefits of additional policy insights. But at least the possibility exists and could be justified for specific policy issues.

Background

Partial Equilibrium (PE) and Computable General Equilibrium (CGE) models are the focus of this paper. Although these models are simplifications of the real world, they provide the policy analyst with a laboratory for testing ideas and policy proposals (McCarl and Spreen 1980; Hazell and Norton 1986). PE and CGE models have strengths and weaknesses for informing judgements about specific policy issues. The choice of modelling framework depends partly on the policy questions under consideration and the nature of the sector(s) to which the policies have substantial impacts on.

Partial equilibrium models

Partial equilibrium models (implemented using mathematical programming frameworks) have been used widely, particularly where industry-specific or technological details are fundamental to the analysis of policy issues. PE models can incorporate detailed economic, policy, regulatory, institutional and technological/biological/engineering characteristics. The inclusion of these attributes in applied economic equilibrium models can provide important insights into policy by creating a link between theory and empirics (McCarl and Spreen 1980; Hazell and Norton 1986). PE models have been applied widely to analyse policy and resource allocation issues in fields such as electricity, natural gas, water, agriculture, environment and trade. Examples are Takayama and Judge 1971; Heady and Srivastava 1975; Meister *et al* 1978; Norton and Solis 1983; Hazell and Norton 1986; Labys *et al* 1989; Heady and Vocke 1992; MacAulay 1992; Salerian *et al* 2000; Barker *et al* 2010; and Loulou *et al* 2016.

A strength of PE models is the scope to model technologies and policies explicitly by drawing on the complementarity (slackness) conditions embodied in mathematical programming. For example, policies that can be binding or non-binding, technologies that can switch on or off or be optimally selected from a portfolio (sometimes endogenously interacting with policy). Consider the Renewable Energy Target operating within the Australian National Electricity Market (NEM). To understand and assess this policy it is desirable to include the appropriate technology and economic principles embodied in peak load pricing and the merit order dispatch of electricity generators, as well as incorporate the policy of a lower bound (floor) on renewable electricity generation target (Murray and Salerian 2018).

The economic equilibrium theory encapsulated in PE models is derived using Karush–Kuhn–Tucker conditions. (Intrilligator 1971; Takayama and Judge 1971).

A limitation of PE models is the inability to incorporate endogenous shifts in income, which are often an economy-wide consequence of policies under investigation (McCarl and Spreen 1980; Hertel 1990; and MacAulay 1992). This specific limitation is overcome using applied CGE models.

Computable general equilibrium models

CGE models have been used widely where inter-industry, inter-regional or economy-wide impacts of policy are important. CGE models used for policy analysis come in a variety of scales and modelling platforms.

Some have evolved over time to become large-scale, recursive comparative static (dynamic) models. These nonlinear models (nonlinear system of simultaneous equations) are often solved by expressing them as linear systems of equations. This is achieved by converting the nonlinear levels models to percentage change form and using multistep solution processes to find a precise solution to the nonlinear system of equations. Examples include the Victoria University Regional Model (VURM, Adams 2015), the United States of America General Equilibrium Model (USAGE, Dixon and Rimmer 2002), and the Global Trade Analysis Project model (GTAP, Hertel 1997).

Other CGE models have been formulated and solved in levels (defined as a nonlinear system of simultaneous equations), as advances in solving nonlinear systems of equations have been developed (examples include Rutherford 1998; Devarajan *et al* 1990; and Lofgren *et al* 2002).

Although CGE models are well-suited to examining sectoral/industry linkages and economy-wide interdependencies, they often have stylised industry representation. For example, technology is often specified using nested constant elasticity of substitution and transformation production functions as illustrated in Lofgren *et al* 2002 (figure 1: Production Technology), Dixon *et al* 1992 (figure E4.5.1: The Input-Activity Specification in the DMR model), and Horridge *et al* 1993 (figure: 6 Structure of production, figure 7: Structure of Investment Demand and figure 8: Structure of Consumer Demand). This is convenient for generating models in the percentage change form, but can be a limitation in capturing the principles underpinning the behaviour of agents in a specific sector.

The economic equilibrium theory encapsulated in CGE models is derived using classical mathematical programming sub-models regarding the behaviour of the various agents in the CGE model of the economy. Constraints are equations, and the theory of the Lagrangean is used to derive the first order equations for a solution (square system of equality constraints and variables).

CGE models with inequality constraints and complementarity conditions

Combining the strengths of both PE and CGE models into a unified modelling framework has proved challenging. Set out below are the three main approaches to solving CGE models with complementarity conditions (inequality constraints and nonnegative variables).

Mixed Complementarity Programming (MCP)

A MCP, as described by Ferris (2000), adds a combinatorial twist to the classic square system of nonlinear equations, thus enabling a broader range of situations to be modelled. These problems arise in a variety of disciplines including engineering and economics where we might want to compute a Wardropian and Walrasian equilibria, and optimisation where we can model the first order optimality conditions for nonlinear programs. Other examples, such as bimatrix games and options pricing, abound.

Examples of developing CGE models using the MCP framework include Rutherford (1987) and Lofgren and Robinson (1997). These CGE models are solved using MCP solvers such as PATH (Dirkse and Ferris 1995) and MILES (Rutherford 1993, 1995).

Mathematical Programming with Equilibrium Constraints (MPEC)

MPEC is a solution strategy that uses mathematical programming methods to solve a CGE model formulated as a MCP. The approach is to transform the MCP problem into Mixed Integer Programming (MIP) or a NonLinear Programming (NLP) models, which are solved iteratively to find a solution to the underlying MCP (CGE) model. Examples include the NLPEC procedure in GAMS (Ferris *et al* 2005) and MPEC procedure in Pyomo (Hart and Siirola 2015).

Sequential Joint Maximisation (SJM)

The SJM approach involves solving a sequence of partial equilibrium relaxations of the underlying general equilibrium model (Dixon 1975; Rutherford 1999), such that the prices and quantities iteratively converge to yield a solution to the CGE model. The partial equilibrium sub models can be solved as MCP or NLP problems.

In this paper, we set out another way to solve CGE models that have inequality constraints and complementarity conditions. The approach is an extension of the Net Social Revenue (NSR) formulation of the spatial and temporal equilibrium modelling framework of Takayama and Judge (1971) and further developments to their framework, which are set out in MacAulay (1992). By using the NSR approach, we can build and solve a CGE model as a single NLP model.

Takayama and Judge Net Social Revenue approach to Economic Equilibria

To understand how we came to the idea that a CGE model could be formulated as a NLP model with economic equilibria described by inequality constraints[1] we need to go back in history. In 1971, Takayama and Judge published Spatial and Temporal Price and Allocation Models. The approach to modelling economic equilibria in spatial markets for the multi-commodity non-integrable case is set out on pages 121 to 125 of their book.

The economic environment is where the regional demand and supply functions do not satisfy the integrability conditions. This means that a welfare objective function cannot be constructed and the approach of solving the spatial equilibrium model using a welfare maximum has to be abandoned.

[1] There are many examples where CGE models that are defined as a simultaneous set of nonlinear equations are solved using nonlinear programming packages, see for example Lofgren et al (2002) page 44. In these situations, the nonlinear programming algorithm's search for a feasible solution is used to solve a nonlinear set of equations. The choice of the objective function has no bearing on the solution as there is only one feasible solution to the model.

However, Takayama and Judge set out an alternative (general) model formulation to solve spatial equilibrium models whether the integrability conditions hold or not. This formulation is called the Net Revenue Maximum (NRM) model. Takayama and Judge state that the welfare maximisation formulation is a special case of the NRM formulation (page 39). Other names used to describe the NRM model are Primal-Dual (page 139), Net Social Monetary Gain (page 250), the General Formulation (Takayama and Labys 1986; MacAulay 1992) and NSR (Takayama and MacAulay (1989 and 1991). We adopt the NSR terminology because we think it best describes the nature of the objective function (discussed later).

Since 1971, there have been significant developments in the application mathematical programming to PE models. Takayama and MacAulay (1991) and MacAulay (1992) show that the original quadratic programming formulation can be generalised to nonlinear functions with desirable properties. MacAulay (1992) also reports on how the NSR formulation can be extended to cover a wide range of policy and economic environments (including non-competitive spatial equilibrium models).

Computer packages and computing power have progressed markedly so that it is now possible to build and solve large scale nonlinear and complementarity programming models.

To begin, we start with the NSR formulation set out Takayama and Judge (1971) described by equation 6.3.1 on page 121 and equation 6.3.2 on page 122 of their book. These are reproduce here as equations 1 and 2. We have added the complementarity variables and conditions for constraints. The linear algebra orthogonality symbol ($\perp$) means that at least one of the adjacent inequalities must be satisfied as an equality (Ferris and Munson 2000, page 167). This also represents the complementarity (slackness) conditions encapsulated in the Karush-Kuhn-Tucker conditions, whereby the complementarity variable can be greater than or equal to zero if the inequality constraint is binding, but is zero if the constraint is not binding. We use this notation to draw out points and observations about the model relevant to developing a way forward for CGE models.

The objective function for the NSR model is given by equation 1, which maximises:

- revenue from sales of commodities in each regional market, given by multiplying the quantity demanded by the Marshallian (indirect) demand function
- less the cost of supplying each commodity in each region, given by multiplying the quantity supplied by the Marshallian (indirect) supply function
- less the cost of transport for each commodity from each supply region to each demand region, given by multiplying the quantity transported by the unit cost of transport.

The symbols i and j refer to regions and k refers to commodities.

$$NR = \sum_i \sum_k y_i^k d_i^k(y_i) - \sum_i \sum_k x_i^k s_i^k(x_i) - \sum_i \sum_j \sum_k t_{ij}^k x_{ij}^k$$

y_i^k is the quantity demanded of commodity k in region i

$d_i^k(y_i)$ is the Marshallian linear demand function for commodity k in region i

x_i^k is the quantity supplied of commodity k in region i (1)

$s_i^k(x_i)$ is the Marshallian linear supply function for commodity k in region i

x_{ij}^k is the quantity transported of commodity k from region i to region j

t_{ij}^k is the unit cost of transport for commodity k from region i to region j

The constraints to the NSR model are given by equation 2.

(a) $d_i^k(y_i) - \rho_i^k \le 0 \quad \perp \quad y_i^k \ge 0$

(b) $-s_i^k(x_i) + \rho^{ik} \le 0 \quad \perp \quad x_i^k \ge 0$

(c) $\rho_i^k - \rho^{ik} - t_{ij} \le 0 \quad \perp \quad x_{ij}^k \ge 0$

(d) $y_i^k - \sum_j x_{ij}^k \le 0 \quad \perp \quad \rho_i^k \ge 0$ (2)

(e) $-x_i^k + \sum_j x_{ij}^k \le 0 \quad \perp \quad \rho^{ik} \ge 0$

(f) $y_i^k \ge 0, x_i^k \ge 0, x_{ij}^k \ge 0, \rho_i^k \ge 0, \rho^{ik} \ge 0$ for all i, j, k

ρ_i^k is the price of demand, ρ^{ik} is the price of supply,

The heart of the formulation is a set of economic equilibrium conditions (equation 2) expressed as inequalities with complementarity variables and conditions. As noted in Takayama and MacAulay (1989), there are two sets of inequality constraints. One set relates to the balance of supply and demand quantities. The other is a set of spatial price equilibrium conditions. The two sets of equilibrium conditions also describe a MCP model, more precisely a linear complementarity programming model, as noted by Takayama and Labys (1986). The equilibrium model described solely by equation 2 can be solved directly using a MCP solver, such as PATH.

But we are interested in the objective function to be used in conjunction with the constraints so that the model can be solved as a NLP model. The question is how to derive the objective function. Where the welfare objective function exists (so that we can start with a pre-existing primal mathematical programming model) it can be shown that the NSR objective function is derived by taking the original (primal) welfare maximising objective function and subtracting from this its dual objective function using the duality formula set out on page 20 of the book. The optimisation model described by equations 1 and 2 can be thought of as the primal-dual model, as set out on pages 138-139 of Takayama and Judge 1971. It has an objective function that is the primal minus the dual and the constraints are those from both The primal and its dual. The constraints in equation 2 are also those from the Karush-Kuhn-Tucker first order conditions to the original welfare maximising primal model.

The NSR objective function can be derived directly from the complementarity programming model defined by equation 2. This is illustrated drawing on:

- the primal problem defined by equations 2.1.1 and 2.1.2 on page 12 of Takayama and Judge (1971) and Balinski and Baumol (1968) equation 1 on page 237
- the dual to 2.1.1 and 2.1.2 as specified by equations 2.1.15 and 2.1.16 on page 17 of Takayama and Judge (noting the typo) and Balinski and Baumol equation 2 on page 237
- subtracting the dual objective function from the primal objective function to derive the primal-dual objective function.

the primal problem is given by equation 3 using the Balinski and Baumol notation, which is less compact than that of Takayama and Judge and is generalised to nonlinear programming.

$$\begin{array}{ll} \text{Maximise} & f(x_1,\dots,x_n) \\ \text{Subject to} & \mathrm{g}_1(\mathrm{x}_1,\dots,x_n) \le \mathrm{c}_1 \quad \perp \quad \rho_1 \ge 0 \\ & \quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ & \mathrm{g}_\mathrm{m}(\mathrm{x}_1,\dots,x_n) \le \mathrm{c}_m \quad \perp \quad \rho_m \ge 0 \\ & x_1,\dots,x_n \ge 0 \end{array} \tag{3}$$

The dual problem is given by equation 4.

$$\begin{array}{ll} \text{Minimise} & f(x) + \sum_i \rho_i [c_i - g_i(x)] - \sum_j x_j \left[\frac{\partial f}{\partial x_j} - \sum_i \rho_i \frac{\partial g_i}{\partial x_j} \right] \\ \text{Subject to} & \rho_1 \frac{\partial g_1}{\partial x_1} + \dots \rho_m \frac{\partial g_m}{\partial x_1} \ge \frac{\partial f}{\partial x_1} \quad \perp \quad x_1 \ge 0 \\ & \quad \vdots \qquad\qquad \vdots \qquad \vdots \\ & \rho_1 \frac{\partial g_1}{\partial x_n} + \dots \rho_m \frac{\partial g_m}{\partial x_n} \ge \frac{\partial f}{\partial x_n} \quad \perp \quad x_n \ge 0 \\ & \rho_i \ge 0 \end{array} \tag{4}$$

The primal-dual model is given by equation 5.

$$
\begin{aligned}
\text{Maximise}\quad & -\sum_i \rho_i\left[c_i - g_i(x)\right] + \sum_j x_j\left[\frac{\partial f}{\partial x_j} - \sum_i \rho_i \frac{\partial g_i}{\partial x_j}\right] \\
\text{Subject to}\quad & \mathrm{g}_1(\mathrm{x}_1,\ldots,x_n) \le \mathrm{c}_1 \quad \perp \quad \rho_1 \ge 0 \\
& \quad\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
& \mathrm{g}_\mathrm{m}(\mathrm{x}_1,\ldots,x_n) \le \mathrm{c}_m \quad \perp \quad \rho_m \ge 0 \\
& \frac{\partial f}{\partial x_1} - \rho_1 \frac{\partial g_1}{\partial x_1} - \ldots - \rho_m \frac{\partial g_m}{\partial x_1} \le 0 \quad \perp \quad x_1 \ge 0 \\
& \quad\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
& \frac{\partial f}{\partial x_n} - \rho_1 \frac{\partial g_1}{\partial x_n} - \ldots - \rho_m \frac{\partial g_m}{\partial x_n} \le 0 \quad \perp \quad x_n \ge 0 \\
& \mathrm{x}_\mathrm{j}, \rho_i \ge 0
\end{aligned}
\qquad (5)
$$

The point to note is that the primal-dual objective function only has terms contained in the complementarity conditions of the constraints to the primal-dual model. This can also be seen by inspecting the net social revenue model presented in Takayama and Judge 1971 (problems 7.2 on page 139, 12.3 on page 250 and 13.1 on page 260) and the various models presented in MacAulay (1992) (problems GM1 to GM5). Therefore, the objective function is derivable solely from the constraints using their complementarity conditions.

The objective function for the MCP model described by the constraints to the NSR model can be derived even without reference to an original single primal model. This is the case for the model (problem 13.1) in Takayama and Judge (1971) and GM3 to GM5 in MacAulay (1992).

To derive the NSR objective function and NLP formulation of the MCP model, we undertake the following steps.

1. Express all constraints as less than or equal to. This is achieved by multiplying greater than or equal to constraints by minus one. We do this to generate a consistent objective function representative of the 'primal minus dual' or net social revenue.
2. Derive the objective function by:
   (a) Multiplying all constraints by their respective complementarity variables
   (b) Bringing all terms to the left hand side
   (c) Summing all of the terms across all of the constraints

The NSR model has the following properties (Takayama and Judge 1971, page 125) at the optimum solution:

- the value of the objective function is zero
- the value of the new Lagrangean variables associated with the constraints in equation 2 must be identical to their respective original complementarity variables in equation 2.

This model is a square system of variables and inequality constraints and there is full symmetry of the Lagrangean variables for constraints with the variables of the model.

Extension to solve CGE models

Let's characterise a CGE model formulated as a MCP model in the following way. There are a set of inequality constraints that define the underlying technologies of production, household consumption and commodity balances in the system. These are the sets of inequalities that define the physical production and consumption of commodities (the physical flows or linkages in our economic system). For example:

- a nested structure of production of composite primary factors, intermediate commodities, domestic commodities, export commodities, import commodities, final composite commodities, investment, household consumption
- an industry technology model (electricity plant investment and operation (merit order dispatch) and transmission investment and operation.

Using our MCP framework, the 'quantity' constraints define complementarity 'price' variables, as set out in equation 6.

$$\begin{array}{llll} g_1(x) = g_1(x_1, \ldots, x_n) - c_1 \le 0 & \perp & v_1 \ge 0 \\ \vdots & & \vdots \\ g_m(x) = g_m(x_1, \ldots, x_n) - c_m \le 0 & \perp & v_m \ge 0 \end{array} \quad (6)$$

Using mathematical programming theory and the Karush-Kuhn-Tucker conditions for optimality, the economic price rules for an equilibrium with respect to each activity (x_j) are based on linking the complementarity price variables down the column of the variable across the constraints, remembering that a CGE model is derived by joining together the equilibrium conditions for the various agents in the economy. Using our MCP framework, these equilibrium 'price' constraints define the complementarity 'quantity' variables, as set out in equation 7. We use the function h(x) to represent any exogenous function representing behaviour of an agent in the CGE model. For example, h(x) could represent the Marshallian foreign export demand function (using the quantity formulation, as introduced in Takayama and Judge 1971, chapter 7). The remaining terms in such a constraint would represent the domestic marginal cost of supplying the export commodity.

$$\begin{array}{lll} h_1(x) - v_1\, \partial g_1/\partial x_1 - \ldots - v_m\, \partial g_m/\partial x_1 \le 0 & \perp & x_1 \ge 0 \\ \vdots & & \vdots \\ h_n(x) - v_1\, \partial g_1/\partial x_n - \ldots - v_m\, \partial g_m/\partial x_n \le 0 & \perp & x_n \ge 0 \end{array} \quad (7)$$

Now note the similarities between equations 6 and 7 and the NSR model of Takayama and Judge (the complementarity constraints in equation 2). In equation 2, constraints (a), (b) and (c) correspond to equation 7. Constraints (d) and (e) correspond to equation 6.

We can derive a NSR objective function for our CGE model in same way. The objective function is given by equation 8.

$$\text{Max NSR} = \sum_{i}^{m} v_i \left[g_i(x) - c_i \right] + \sum_{j} x_j \left[h_j(x) - \sum_{i}^{m} v_i \, \partial g_i / \partial x_j \right] \qquad (8)$$

Symmetry of the NLP variables of the CGE model and the new Lagrangean variables

To illustrate why the NLP version of the CGE model has symmetry of its Lagrangean and CGE variables, we apply the Karush-Kuhn-Tucker first order conditions to examine the equilibrium conditions for our new optimisation model. This mirrors the approach outlined in Takayama and MacAulay (1989, page 6) and MacAulay 1992 (pages 302-303). We first create the Lagrangean function with two new sets of Lagrangean variables corresponding to the two sets of equilibrium constraints in our initial CGE model expressed as an MCP model.

$$\begin{aligned} \text{Max L} = & \sum_{i}^{m} v_i \left[g_i(x) - c_i \right] + \sum_{j}^{n} x_j \left[h_j(x) - \sum_{i}^{m} v_i \, \partial g_i / \partial x_j \right] \\ & + \sum_{i}^{m} \varphi_i \left[c_i - g_i(x) \right] + \sum_{j}^{n} \gamma_j \left[-h_j(x) + \sum_{i}^{m} v_i \, \partial g_i / \partial x_j \right] \\ & \text{where } x_j \geq 0, v_i \geq 0, \varphi_i \geq 0, \gamma_j \geq 0 \end{aligned} \qquad (9)$$

The first order conditions are given by the following equations.

$$\frac{\partial L}{\partial v_i} = g_i(x) - c_i - \sum_{j}^{n} x_j \, \partial g_i / \partial x_j + \sum_{j}^{n} \gamma_j \, \partial g_i / \partial x_j \leq 0 \quad \perp \quad v_i \geq 0 \quad \text{for } i = 1, \ldots, m \qquad (10)$$

$$\frac{\partial L}{\partial \varphi_i} = g_i(x) - c_i \leq 0 \quad \perp \quad \varphi_i \geq 0 \quad \text{for } i = 1, \ldots, m \qquad (11)$$

$$\begin{aligned} \frac{\partial L}{\partial x_j} = & \sum_{i}^{m} v_i \, \partial g_i / \partial x_j + x_j \left[\partial h_j / \partial x_j - \sum_{i}^{m} v_i \, \partial^2 g_i / \partial x_j^2 \right] + \left[h_j(x) - \sum_{i}^{m} v_i \, \partial g_i / \partial x_j \right] \\ & - \sum_{i}^{m} \varphi_i \, \partial g_i / \partial x_j - \gamma_j \left[\partial h_j / \partial x_j - \sum_{i}^{m} v_i \, \partial^2 g_i / \partial x_j^2 \right] \leq 0 \quad \perp \quad x_j \geq 0 \\ & \text{for } j = 1, \ldots, n \end{aligned} \qquad (12)$$

$$\frac{\partial L}{\partial \gamma_j} = h_j(x) - \sum_{i}^{m} v_i \, \partial g_i / \partial x_j \leq 0 \quad \perp \quad \gamma_j \geq 0 \quad \text{for } j = 1, \ldots, n \qquad (13)$$

For brevity and consistency, we write the Karush-Kuhn-Tucker complementarity conditions using the MCP notation. Equations 11 and 13 are the same as those in our original CGE except that they now define the new Lagrangean (complementarity) variables for the constraints of our new nonlinear programming problem. At the optimum solution to our new nonlinear programming problem, $v_i = \varphi_i$ and $x_j = \gamma_j$. This is the same as for the NSR models of Takayama and Judge, as noted in MacAulay (1992, equation 9 on page 303).

With $v_i = \varphi_i$ and $x_j = \gamma_j$ at the optimum solution, terms cancel out in equations 6 and 8 and they collapse to the original constraints in the CGE model. Therefore:

- the value of the objective function is zero
- the original constraints and complementary conditions hold
- the solution is symmetrical with matching pairs of the new Lagrangean variables and the variables in the original CGE model.

The MCP solution of the CGE model using PATH also exhibits this symmetry.

Concavity of the maximisation objective function

In a standard CGE model most production functions are constant returns to scale. So what makes our maximisation objective function concave and the model bounded? In our single country, long run static CGE model, it is the fixed aggregate supply of labour in the economy and the downward sloping Marshallian export demand functions. This property is often employed in standard CGE models and explains why export demand functions are price responsive.

Interpretation of the objective function

The objective function as derived must be zero based on the Karush-Kuhn-Tucker complementarity conditions applying to the constraints. Multiplication of each inequality constraint by its complementarity variable must be zero and therefore summation of these constraints must be zero.

However, there is an economic interpretation to the objective function. The objective function is analogous to that of Social Accounting Matrices (Gilbert and Tower 2013, chapter 25) used to calibrate CGE models.

One can think of the set of commodity flow constraints multiplied by their respective price (complementarity) variables as the value of flows of goods and services (sources and uses) in the economy (which correspond to rows of a social accounting matrix).

We can think of the equilibrium price rules multiplied by their respective quantity (complementarity) variables as the values of receipts and payments for agents in the economy (which correspond to the columns of a social accounting matrix).

It is for this reason we prefer to use the term net social revenue to describe the objective function.

The objective function should not be confused with zero net profit to society, driven by competitive market rules governing consumption, production and resource allocation. The equilibrium models can include many forms of monopolistic and monopsonistic economic behaviour by agents in the economy (which create rents), as illustrated in MacAulay (1992). However, any economic rents do have to be correctly accounted for in transactions.

The properties of the model are useful for developing and calibrating a model and verifying that it is correctly specified. Make a mistake and the objective function will not be zero and there will not be symmetry of the solution.

Applied Examples

For illustrative purpose, the GAMS (GAMS 2019) code for a small standard CGE model is provided *here* (a *link to the GAMS model code and solution list file on the journal website*). The model is based on that published by Hosoe (2004) in the GAMS model library. This model is small enough to be solved using the free demonstration version of GAMS model building software and two solvers available in GAMS (PATH used to solve the MCP and CONOPT used to solve the NLP), so anyone can download the GAMS software and run these models. We use these models to illustrate the how the NLP version of a CGE model is applied in practice and to demonstrate that the approach yields identical solutions to the MCP version.

We have made some changes to the Hosoe model to make it representative of a long run steady state economy. We do this by linking industry investment explicitly to the capital stock in each industry. The modifications to the model include:

- exogenously fixing the domestic ownership share of capital stock
- endogenising domestic saving
- creating and linking investment activities for each industry to their capital stocks.

We solve both the MCP (using PATH) and NLP (using CONOPT) models for three scenarios

- a base case model (which gives us the initial CGE solution), which is also one of the tests for a valid model in levels form set out in Dixon and Rimmer (2013)
- we check that the nominal homogeneity test holds for our single-country model in which the nominal exchange rate is the numeraire price, which is another test for a valid model CGE model (Dixon and Rimmer 2013)
- we conduct a policy experiment (a cut in company tax of 10 per cent)

The MCP and NLP models yield identical results.

Elsewhere, we have applied the approach to illustrate how we can combine an electricity sector PE (Salerian *et al* 2000) model with a traditional CGE for Australia based on the CGE framework of the smaller GAMS model we have provided with this paper (Murray and Salerian 2018; Productivity Commission 2019). This model is used to illustrate the economic principles underlying the Australian National Electricity Market (NEM) and the Renewable Energy Market by including Australia's Renewable Energy Target (RET) framework explicitly. A description of this model, including the GAMS code is also available from the Productivity Commission website.

Conclusion

Solving CGE models with inequality constraints and complementarity conditions using a unifying nonlinear programming framework opens the door to a wider range of ways to build and solve such models. The NSR (and NLP) approach set out here is more direct and simpler than the SJM and MPEC approaches. There are now many packages available for building and solving nonlinear programming models, including open source model building languages and solvers, such as Pyomo/Ipopt (Hart *et al* 2012; COIN-OR foundation 2019).

We have not assessed the comparative efficiency of building and solving CGE models using the Net Social Revenue formulation versus the alternatives. This is an area for future research.

We do see scope to broaden the types of CGE models developed. For example, risk could be introduced into CGE models using the state contingent framework of Chambers and Quiggin (2000). This could be used to model risk regarding generation supply from renewable sources and the use of storage batteries to arbitrage generation across time periods reflecting the state of demand and supply. This a natural extension of the temporal price models set out in Takayama and Judge (1971).

Such approaches will not be appropriate for every exercise. For some analyses, the cost of the additional effort and time required to build a complete, integrated model can exceed the additional benefits gained. But at least the possibility exists and could be justified for specific policies.

References

Adams, P., (2015) *The Victoria University Regional Model (VURM): Technical Documentation*, Melbourne: Victoria University, accessed online: http://www.copsmodels.com/ftp/workpapr/g-254.pdf.

Barker, A., Murray, T. and Salerian, J. (2010), *Developing a Partial Equilibrium Model of an Urban Water System*, Productivity Commission Staff Working Paper, Melbourne, March.

Balinski, M. L. and Baumol, W. J. (1968) 'The dual in Nonlinear Programming and its Economic Interpretation', *Review of Economic Studies*, vol. 35, issue 3, pp. 237–256.

Chambers, R. G. and Quiggin, J. (2000) *Uncertainty, production, choice and agency: the state-contingent approach*, Cambridge University Press, New York.

COIN-OR Foundation (2019) (accessed at https://www.coin-or.org/about-the-foundation/ on 07/01/2019).

Devarajan, S., Lewis, J., and Robinson, S. (1990) 'Policy Lessons from Trade Focused, Two-Sector Models', *Journal of Policy Modeling*, 12, pp. 625–657.

Dirkse, S., and Ferris, M. (1995), 'The PATH Solver: A Non-Monotone Stabilization Scheme for Mixed Complementarity Problems', *Optimization Methods and Software*, 5, pp. 123–156.

Dixon, P. (1975) *The Theory of Joint Maximization*, North-Holland Publishing Company, Amsterdam.

Dixon, P., and Jorgenson, D. (2013), *Handbook of Computable General Equilibrium Modeling*, 2 volumes, Amsterdam: Elsevier.

Dixon, P., Parmenter, B. R., Powell A. A. and Wilcoxen, P. J. (1992) Notes and Problems in Applied General equilibrium Economics, North Holland, Amsterdam.

Dixon, P., and Rimmer, M. (2002) *USAGE-ITC: Theoretical Structure*, Centre of Policy Studies, Victoria: Monash University.

Dixon, P. B., Rimmer, M. T., (2013) *Validation in Computable General Equilibrium Modeling*. in: Dixon, P.B., Jorgenson, D. W. (Editors.), Handbook of Computable General Equilibrium Modeling. North Holland, Elsevier B. V., pp. 1271–1330.

Ferris, M., Dirkse, S. and Meeraus, A. (2005) 'Mathematical Programs with equilibrium Constraints: Automatic Reformulation and Solution via Constrained Optimization', in T.J. Kehoe, T.N. Srinvasan and J. Whalley, editors, *Frontiers in Applied General Equilibrium Modelling*, pp. 67–93, Cambridge University Press.

Ferris, M., and Munson, T. (2000) 'Complementarity Problems in GAMS and the PATH Solver', *Journal of Economic Dynamics and Control*, 24, pp. 165–188.

GAMS (2019) (accessed at http://www.gams.com on 07/01/2019).

Gilbert, J. and Tower, E. (2013) *Introduction to numerical Simulation for trade theory and policy,* World Scientific Publishing, Singapore.

Hart, W. E., Laird, C., Watson, J.-P. and Woodruff, D. L. (2012) 'Pyomo — Optimzation Modeling in Python', *Springer Optimization and Its Application*, Volume 67, Springer, New York.

Hart, W. E. and Siirola, J. D. (2015) *Modelling mathematical Programs with equilibrium Constraints in Pyomo*, SAND2015-5584, Sandia National Laboratories.

Horridge, J. M., Parmenter, B. R. and Pearson, K. R. (1993) 'ORANI-F: A General equilibrium Model of the Australian Economy', *Economic and Financial Computing*, 3 (2) pp. 71–140.

Hosoe, N. (2004) *A Standard CGE Model*, Seq: 276, Name: stdcge (accessed at https://www.gams.com/latest/gamslib_ml/libhtml/index.html on 04/01/2019).

Lofgren, H. and Robinson, S. (1997) *The Mixed-Complementary Approach to Specifying Agricultural Supply in Computable General Equilibrium Models*, TMD discussion papers 20, International Food Policy Research Institute (IFPRI).

Hazell, P. and Norton R. (1986) *Mathematical Programming for Economic Analysis in Agriculture*, New York, NY: MacMillan.

Heady, E., and Vocke, G. (eds) (1992), *Economic Models of Agricultural Land Conservation and Environmental Improvement*, Ames, USA: Iowa State University Press.

Heady, E., Srivastava, U. (1975) *Spatial Sector Programming Models in Agriculture*, Ames, USA, Iowa State University Press.

Hertel, T. (1997) *Global Trade Analysis: Modeling and Applications*, London, Cambridge University Press.

Hertel, T., (1990) 'General Equilibrium Analysis of U.S. Agriculture: What Does It Contribute', *The Journal of Agricultural Economics Research*, 42(3), pp. 3–9.

Intriligator M. (1971) *Mathematical Optimization and Economic Theory*, New Jersey: Prentice-Hall.

Labys, W., Takayama, T. and Uri, N. (1989) *Quantitative Methods for Market Oriented Economic Analysis Over Space and Time*, London: Gower Publishing Company.

Lofgren, H., Harris, R. L., Robinson, S., Thomas, M. and El-Said, M (2002) *A Standard Computable General Equilibrium (CGE) model in GAMS*, Microcomputers In Policy Research 5, International Food Policy Research Institute.

Loulou, R., Lehtila, A., Kanudia, A., Remme, U. and Goldstein, G. (2016) *Documentation for the TIMES Model: PART II*, *Energy Technology Systems Analysis Programme* (accessed at http://www.iea-etsap.org/web/Documentation.asp on 02/01/2019).

MacAulay, G. (1992) 'Alternative Spatial Equilibrium Formulations: A Synthesis', in Griffiths, W., Lutkepohl, H., and Bock, M. (1992) *Readings in Economic Theory and Practice*, Chapter 13, Amsterdam: North-Holland, pp. 297–326.

McCarl, B. and Spreen, T.H. (1980) 'Price Endogenous Mathematical Programming as a Tool for Sector Analysis", *American Journal of Agricultural Economics*, 62(1), pp. 87–102.

Meister, A., Chen, C., and Heady, E. (1978) *Quadratic Programming Models Applied to Agricultural Policies*, Ames, USA: Iowa State University Press.

Murray, T. and Salerian, J. (2018) Unifying partial and general equilibrium modelling for applied policy analysis, presentation at the Australian Conference of Economists, 10-13 July 2018, Canberra.

Norton, R., and Solis, L. (1983) *The Book of CHAC: Programming Studies for Mexican Agriculture*, Baltimore: John Hopkins University Press.

Rutherford, T. (1987) *Applied general equilibrium modelling, Ph.D thesis*, Department of Operations Research.

—— (1998) *GTAPinGAMS: The Dataset and Static Model* (accessed at https://www.gtap.agecon.purdue.edu/resources/download/64.pdf on 03/01/2019).

—— (1993) *MILES: a Mixed Inequality and non-Linear Equation Solver*, Working Paper, Department of Economics, University of Colorado.

—— (1995) 'Extension of GAMS for Complementarity Problems Arising in Applied Economic Analysis', *Journal of Economic Dynamics and Control*, 19(8), pp 1299–1324.

—— (1999) 'Sequential Joint Maximization', in Weyant, J. (editor) *Energy and Environmental Policy Modeling. International Series in Operations Research and Management*, vol. 18. Springer, Boston, MA.

Salerian, J., Gregan, T. and Stevens, A. (2000) 'Pricing in Electricity Markets', *Journal of Policy Modeling*, 22(7), pp. 859–893.

Takayama, T., and Judge, G. (1971) *Spatial and Temporal Price Allocation Models*, Amsterdam: North-Holland Publishing.